ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 10, pp. 995–1015.

TOPICAL ISSUE

Spectral Decompositions of Inverse Gramian Matrices and Energy Metrics of Continuous Dynamic Systems

I. B. Yadykin

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: Jad@ipu.ru

Received May 31, 2024
Revised July 15, 2024
Accepted July 25, 2024

Abstract—The article is aimed at developing new algorithms for element-by-element calculation of matrices of direct and inverse gramians for stable continuous linear MIMO LTI systems based on spectral decompositions of gramians in the form of Hadamard products. It is shown that the multiplier matrices in the Hadamard product are invariant under various canonical transformations of linear continuous systems. Spectral decompositions of inverse matrices of gramians of continuous dynamic systems from the spectra of gramian matrices and the original dynamics matrices are also obtained. The properties of the multiplier matrices in spectral decompositions of gramians are studied. Using these results, spectral decompositions of the following energy metrics were obtained: of the volumes of attraction ellipsoids, of the matrix traces of direct and inverse controllability gramians, of the input and output system energies, of the centrality indices of energy controllability metrics and of the average minimum energy. The practical applicability of the results is considered.

Keywords: spectral decompositions, continuous dynamical systems, gramians, Lyapunov equations, multiplier matrices, Hadamard product

DOI: 10.31857/S0005117924100074

1. INTRODUCTION

Gramian matrices are solutions of Lyapunov and Sylvester equations of a special type, which are currently quite well studied [1–7]. Although control theory offers mathematical tools for controlling technical and natural systems, the scientific foundations for controlling complex cyber-physical systems are still insufficiently developed. In [8], analytical tools were proposed for studying the controllability of an arbitrary complex directed network based on the optimal choice of control nodes of the network, which can effectively control the entire dynamics of the system. Application of these tools to real networks revealed that the number of control nodes is determined mainly by the distribution of network nodes. Control theory is a mathematically advanced branch of technology, which has numerous applications to electrical systems, industrial processes, communication systems, aircraft, and spacecraft. However, fundamental questions concerning the controllability of complex systems arising in nature and technology have not yet been fully resolved. Using a controllability metric defined as the proportion of effectively controlling network nodes in their minimum set required for complete controllability, it is shown in [9] that sparse heterogeneous networks have poor controllability, while dense homogeneous networks have better controllability in comparison. Transformation of the equations of state into canonical forms of controllability and observability allows one to simplify the solution of the Lyapunov equations and to study the structural properties of controllability and observability [9–12]. An important problem of optimal placement of sensors and actuators based on various energy functionals, including invariant ellipsoids, was considered in [13–16]. In [10], a general approach to solving the problem of optimal placement of sensors and actuators for multivariate control systems was formulated, which is based on the decomposition of the system into stable and unstable subsystems. The degree of controllability of the system is determined on the basis of energy metrics based on the use of finite and infinite controllability gramians. A general method for calculating the inverse controllability gramian for the equations of state specified in canonical controllability forms is proposed. In [13], a method for optimal placement of virtual inertia on the graph of an energy system is proposed, it is based on the use of energy metrics of the coherence of generators and the square of the H_2-norm of the system operator, specified by the standard dynamic model in the state space. The problem is formalized as a non-convex optimization problem with constraints in the form of values of observability gramians. It is well known that minimum energy control problems are also solved using gramians. In recent years, these approaches have been developed for complex energy, social, transport and biological networks in [17–21]. In [19], it was shown that in many cases, the closer the eigenvalues of the dynamics matrix are to the imaginary axis, the less energy is required to ensure full controllability of the network. Thus, the degree of controllability (reachability) of the network is associated with the minimum energy, which allows one to introduce new metrics in the form of the minimum eigenvalue of the controllability gramian and the maximum number of its inverse gramian, as well as traces of these gramians. In the power industry, DC inserts are used to damp dangerous low-frequency oscillations. Using the gramian method for optimal placement of DC inserts in a full-scale model of the European power system, it was possible to successfully solve the problem of global optimization of the placement of DC inserts on the graph of the system model.

2. PRELIMINARY DISCUSSION AND PROBLEM STATEMENT

A directed graph G, formed by a set of nodes E and a set of edges Q is considered. A linear dynamic graph model with a standard description in the form of $(A,\ B,\ C)$ representation in the state space can be used to describe the graph model. As such a model, a stable linear stationary continuous dynamical system with many inputs and many outputs is considered

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = 0, \tag{2.1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^m$.

Large dynamic networks can be described by equations (1), where A is the Laplacian matrix, B is the matrix of control inputs, and C is the matrix of measured outputs and the order of the graph dynamics matrix is a sufficiently large positive number [19]. The Lyapunov equation for computing the controllability gramian of system (2.1) has the following form

$$AP^c + P^cA^{\mathrm{T}} = -BB^{\mathrm{T}}. \tag{2.2}$$

Each actuator in a real system is limited in its control energy, so an important class of controllability metrics deals with the amount of input energy required to reach a final state from the initial state. The following optimal control problem with minimum energy that will transfer the system from the initial state to the final state x_f at time t [10] can be formulated

$$\underset{u(t)\in\mathcal{L}_2}{minimize} \int\limits_0^{\mathrm{T}} \|u\,(\tau)\|^2 d\tau, \tag{2.3}$$

$$\dot{x}\,(t) = Ax\,(t) + Bu\,(t)\,,$$
$$x\,(0) = 0, \quad x\,(t) = x_f.$$

Standard methods of optimal control theory can be used to obtain the solution. If the system is controllable, then the optimal input $u(\tau)^*$ has the form

$$u(\tau)^* = B^{\mathrm{T}} e^{A^{\mathrm{T}}(\tau - t)} \left(\int_0^{\mathrm{T}} e^{A\sigma} B B^{\mathrm{T}} e^{A^{\mathrm{T}}} d\sigma \right)^{-1} x_f, \quad 0 \leqslant \tau \leqslant t.$$

The optimum minimum of the input energy is

$$\int_0^{\mathrm{T}} \|u(\tau)\|^2 d\tau = x_f^{\mathrm{T}} \left(\int_0^{\mathrm{T}} e^{A\sigma} B B^{\mathrm{T}} e^{A^{\mathrm{T}}} d\sigma \right)^{-1} x_f. \tag{2.4}$$

The average minimum energy is defined by the expression [19]

$$E_{avmin} = \frac{1}{n} \operatorname{tr} P_c^{-1} = \frac{\int_{\|x\|=1} x^{\mathrm{T}} P_c(t) x dx}{\int_{\|x\|=1} dx}.$$

The matrix

$$P_c(t) = \int_0^{\mathrm{T}} e^{A\sigma} B B^{\mathrm{T}} e^{A^{\mathrm{T}}} d\sigma \tag{2.5}$$

is called the finite controllability gramian at time t. The controllability gramian $P_c(t)$ in most cases is a positive semidefinite matrix. It defines an ellipsoid in the state space

$$\mathcal{E}_{\min} = \left\{ x \in R^n \,\middle|\, x^{\mathrm{T}} P_c(t)^{-1} x \leqslant 1 \right\},$$

which contains a set of states that can be reached in the moment t. The eigenvectors and corresponding eigenvalues of the matrix $P_c(t)^{-1}$ determine the lengths of the corresponding semi-axes of the ellipsoid [6].

Definition 1. Xiao matrix is a pseudo-hankel square matrix that has a zero-plaid structure of the form [9]

$$Y = \begin{bmatrix} y_1 & 0 & -y_2 & 0 & y_3 \\ 0 & y_2 & 0 & -y_3 & 0 \\ -y_2 & 0 & y_3 & 0 & \dots \\ 0 & -y_3 & 0 & \dots & 0 \\ y_3 & 0 & \dots & 0 & y_n \end{bmatrix}, \qquad y_i \in \mathsf{C}, \quad i = \overline{1, n}.$$

The elements of the matrix are designated using formulas

$$y_{j\eta} = \begin{cases} 0, \ \text{ if } \ j + \eta = 2k + 1, & k = 1, 2 \dots, n; \\ (-1)^{\frac{j-\eta}{2}} \boldsymbol{y_n}, \ \text{ if } \ j + \eta = 2k, & k = 1, 2 \dots, n. \end{cases}$$

An important special case of continuous linear stationary SISO (single-input, single-output) LTI systems represented by state equations in canonical controllability and observability forms is considered. In this case, the controllability and observability gramians are defined by the formulas [20]

$$P^{cF} = \sum_{k=1}^{n} \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} 1_{j+1\eta+1}, \tag{2.6}$$

$$P^{oF} = \sum_{k=1}^{n} \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} 1_{j+1\eta+1}, \tag{2.7}$$

where $N(s)$ is the characteristic polynomial of the system.

The representation of gramians in Hadamard form according to the formulas (2.6)–(2.7)

$$P^{cF} = \Omega_{cF} \circ \Psi_c, \quad \Psi_c = \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} 1_{j+1\eta+1},$$

$$P^{oF} = \Omega_{oF} \circ \Psi_o, \quad \Psi_o = \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} 1_{j+1\eta+1.}$$

Hence follow the identities

$$P^{cF} \equiv \Omega_{cF} = \sum_{k=1}^{n} \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} 1_{j+1\eta+1}, \tag{2.8}$$

$$P^{oF} \equiv \Omega_{oF} = \Omega_{cF}.$$

The matrices Ω_{cF}, Ω_{oF} are Xiao matrices [20]. An alternative form of presentation is valid

$$\Omega_{cF} = \omega(n, s_k, j, \eta) 1_{j+1\eta+1},$$

$$\omega(n, s_k, j, \eta) = \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)}, \quad j, \eta = 0, n-1.$$

Note that $j+1, \eta+1$ are the row and column numbers of the Xiao matrix. The function $\omega(n, s_k, j, \eta)$ is a scalar multiplier and an invariant under various similarity transformations [7]. A dynamic network described by the equations of state of a graph of the form (2.1) is considered. The matrix V is defined as

$$V = \begin{bmatrix} e_1 & \dots e_i \dots & e_n \end{bmatrix},$$

where e_i is a unit vector in R^n, i is the number of the graph node. Following [21] the centrality index of the energy metric of controllability of a graph node is defined as

$$J_{CE_i} = \operatorname{tr}(P_{ci}),$$

where P_{ci} is the infinite gramian of controllability of a graph node, defined as solutions of the following modal Lyapunov equations

$$AP_{ci} + P_{ci}A^{\mathrm{T}} = -e_i e_i^{\mathrm{T}}.$$

Due to the linearity of the Lyapunov equations, the following statements are true

$$J_{CE} = \sum_{i=1}^{n} J_{CE_i}.$$

This equation defines the centrality index of the energy metrics of the graph. Generalizing these results for the case of continuous MIMO LTI systems, one can introduce a similar centrality index of the energy metrics of controllability of individual modes of these systems for the case of stable fully controllable and observable systems with a simple spectrum of the dynamics matrix

$$J_{CE_i} = \operatorname{tr}(P_{ci}),$$

where i is number of an individual eigenvalue of the dynamics matrix. The rationale for this approach is that any linear continuous MIMO LTI system can be represented as a graph whose nodes correspond to individual eigenvalues. As shown in [10], the following formula is valid

$$P_{ci} = e_i \left(\int_0^\infty e^{A\sigma} BB^{\mathrm{T}} e^{A^*\sigma} d\sigma \right) e_i^{\mathrm{T}} = e_i Q(0, \infty) e_i^{\mathrm{T}},$$

where $Q(0, \infty)$ is the infinite grammian of controllability.

In [10] it is shown that the matrix $Q(0,\infty)$ is a Xiao matrix, which is invariant under any non-degenerate coordinate transformations and a positive-definite matrix. Note that controllability gramians play a central role in the formation of invariant ellipsoids of attraction. Further the research objectives for solving some problems of state estimation and control will be considered. The first objective of the article will be to obtain spectral decompositions of the inverse matrices of gramians of continuous dynamic systems from the spectra of the inverse matrices and the original dynamics matrices. Another objective of this article is to obtain spectral decompositions of the following energy metrics [6, 8–19]:

1) of the volume of the attraction ellipsoids $Vol\epsilon_x, Vol\epsilon_y$. This metric characterizes the volume of a subset of the state space reachable from the origin for a fixed amount of control energy, which is a function of the controllability gramian determinant.

2) of traces of the controllability gramian matrix trP_c. This metric characterizes the measure of energy, which is inversely proportional to the average energy required to control the system in different directions of the state space.

3) of traces of the inverse matrix of the controllability gramian $\mathrm{tr}\,{P_c}^{-1}$. This metric characterizes the energy measure, which is proportional to the average energy required to control the system in different directions of the state space.

4) of the input and output energies of the system $E_{in},\ E_{out}$.

5) of the centrality indices of energy metrics of controllability of the system of individual modes of continuous multi-connected stationary systems $J_{CE},\ J_{CE_i}$.

6) of the average minimum energy E_{avmin}.

3. SEPARABLE SPECTRAL DECOMPOSITIONS OF INVERSE GRAMIANS OF CONTINUOUS MIMO LTI SYSTEMS IN THE FORM OF GENERALIZED XIAO MATRICES

The following two methods for determining spectral decompositions of the inverse matrix of controllability gramians will be proposed below.

First method. Separable spectral decompositions of inverse gramians of continuous MIMO LTI systems (2.1) are based on the decomposition of the resolvent of the controllability gramian matrix in the Faddeev–Leverrier series and the calculation of the zero term of this decomposition. The resolvent decomposition has the form [22, 23]

$$(Is - P_c)^{-1} = \frac{\sum_{j=0}^{n-1} R_{j+1} s^j}{N(s)}. \tag{3.1}$$

Assume $N(s) = s^n + p_{n-1}s^{n-1} + \ldots p_1 s + p_0$, $j = 1, 2, \ldots n$. Denote λ_k the roots of the characteristic equation. The Faddeev matrices R_j and the coefficients of the characteristic equation of the gramian are determined using the following algorithm

The 1st step: $p_n = 1,\ R_n = I,\ R_i = \sum_{j=i}^{n} p_j P_c^{j-i}, \quad i = 1, 2, \ldots, n.$

. . . .

The kth step: $p_{n-k} = -\frac{1}{k}\mathrm{tr}\,(P_c R_{n-k+1}),\ R_{n-k} = p_{n-k}I + P_c R_{n-k+1},\ k = 1, 2, \ldots, n.$

. . . .

The nth step: $p_0 = -\frac{1}{n}\mathrm{tr}\ (P_c R_1),\ R_0 = p_0 I + P_c R_1 = 0.$

Assume $s = 0$ [24]

$$P_c^{-1} = -p_0^{-1} R_1 = p_0^{-1} - p_1 I - p_2 P_c - \ldots - P_c^{n-1}. \tag{3.2}$$

Thus, using the spectral decomposition of the resolvent of the controllability gramian matrix, a formula for calculating the inverse gramian matrix is obtained. Note that not only the inverse gramian matrix is calculated, but also all the coefficients of the characteristic polynomial of the gramian, which allows to calculate all the eigenvalues of the gramian matrix. In addition, for a simple spectrum of the gramian matrix, a formula representing the spectral decomposition of the inverse gramian over the spectrum of the gramian matrix is obtained

$$P_c^{-1} = \frac{\sum_{\lambda=1}^{n} \sum_{j=0}^{n-1} P_{c,j}\sigma_\lambda^j}{\dot{N}_c(\sigma_\lambda)} \frac{1}{\sigma_\lambda},$$

where P_c is the controllability gramian, $P_{c,j}$ is Faddeev matrix in the gramian resolvent decomposition, σ_λ are eigenvalues of gramian matrix P_c.

Second method. From (3.2) it is clear that the inverse matrix is formed by the sum of non-negative powers of the gramian. This observation implies the following result.

Lemma 1. *For MIMO LTI system* (2.1) *with a simple spectrum, the real matrices* A, BB^{T}, *for* A *with a simple spectrum, and distinct eigenvalues* s_k, s_ρ, *the controllability gramian has the structure of the Xiao matrix* [24]:
for even n

$$P_c = \begin{bmatrix} p_{11} & 0 & -p_{22} & \dots & 0 \\ 0 & p_{22} & \dots & 0 & \dots \\ -p_{22} & \dots & \dots & \dots & -p_{n-1n-1} \\ \dots & 0 & \dots & p_{n-1n-1} & 0 \\ 0 & \dots & -p_{n-1n-1} & 0 & p_{nn} \end{bmatrix},$$

for odd n

$$P_c = \begin{bmatrix} p_{11} & 0 & -p_{22} & \dots & p_{\frac{n+1}{2}\frac{n+1}{2}} \\ 0 & p_{22} & \dots & -p_{\frac{n+1}{2}\frac{n+1}{2}} & \dots \\ -p_{22} & \dots & \dots & \dots & -p_{n-1n-1} \\ \dots & -p_{\frac{n+1}{2}\frac{n+1}{2}} & \dots & p_{n-1n-1} & 0 \\ p_{\frac{n+1}{2}\frac{n+1}{2}} & \dots & -p_{n-1n-1} & 0 & p_{nn} \end{bmatrix}.$$

And the inverse matrix of the controllability gramian has the form of
for even n

$$P_c^{-1} = \begin{bmatrix} \tilde{p}_{11} & 0 & \tilde{p}_{13} & \dots & 0 \\ 0 & \tilde{p}_{22} & \dots & 0 & \dots \\ \tilde{p}_{31} & \dots & \dots & \dots & \tilde{p}_{n-2n} \\ \dots & 0 & \dots & \tilde{p}_{n-1n-1} & 0 \\ 0 & \dots & \tilde{p}_{nn-2} & 0 & \tilde{p}_{nn} \end{bmatrix},$$

for odd n *in the form of*

$$P_c^{-1} = \begin{bmatrix} \tilde{p}_{11} & 0 & \tilde{p}_{13} & \dots & \tilde{p}_{1n} \\ 0 & \tilde{p}_{22} & \dots & \tilde{p}_{2n-1} & \dots \\ \tilde{p}_{31} & \dots & \dots & \dots & \tilde{p}_{n-2n} \\ \dots & \tilde{p}_{n-12} & \dots & \tilde{p}_{n-1n-1} & 0 \\ \tilde{p}_{n1} & \dots & \tilde{p}_{nn-2} & 0 & \tilde{p}_{nn} \end{bmatrix}.$$

Such a structure defined as a structure of the generalized Xiao matrix. This matrix inherits from the Xiao matrix the presence of zero in those elements where the sum of the row and column is odd.

Proof. According to (3.1) the structure of the matrix $(P_c)^{-1}$ coincides with the structure of the Faddeev matrix R_1. The latter matrix is a linear combination of non-negative powers of the matrix P_c, which is a sum of zero plaid multipliers [11]. Any positive power "k" of the matrix P_c does not change the structure of the multiplier. Adding a diagonal matrix $p_0^{-1}p_1 I$ to such a structure does not change the structure of the multiplier matrix. The lemma says nothing about calculating the elements of the matrix $(P_c)^{-1}$. This problem can be solved by reducing the problem of calculating the inverse matrix to a set of solutions of n linear systems of algebraic equations with the same left-hand matrix.

Theorem 1. *If the conditions of the lemma be satisfied, then the elements of the inverse gramian matrix can be evaluated by solving a system of linear algebraic equations of the SLAE type*

$$P_c X = I, \tag{3.3}$$

where the matrix P_c is a generalized Xiao matrix. System (3.3) *reduces to the solution of n systems of linear algebraic SLAE equations of the form*

$$P_c x_i = e_i, \tag{3.4}$$

where x_i is column "i" of matrix X, and e_i is a unit vector.

Thus, the described method can be called hybrid, since it combines the decomposition of the resolvent of the controllability gramian matrix in the Faddeev–Leverrier series and the SLAE method. For low-dimensional systems, it allows one to obtain the following calculation formulas for calculating the elements of the inverse gramian matrix in Xiao form.

Illustrative Example

The solutions of the SLAE (3.3) for low-dimensional systems are

$\boldsymbol{n = 1}$.

$$x_{11} = (p_{11})^{-1}.$$

$\boldsymbol{n = 2}$.

$$x_{11} = (p_{11})^{-1}, \;\; x_{22} = (p_{22})^{-1}.$$

$\boldsymbol{n = 3}$.

$$x_{11} = p_{33}\left(p_{11}p_{33} - p_{22}^2\right)^{-1}, \;\; x_{31} = p_{22}\left(p_{11}p_{33} - p_{22}^2\right)^{-1},$$
$$x_{22} = (p_{22})^{-1}.$$
$$x_{13} = p_{22}\left(p_{11}p_{33} - p_{22}^2\right)^{-1}, \;\; x_{33} = p_{11}\left(p_{11}p_{33} - p_{22}^2\right)^{-1},$$

$\boldsymbol{n = 4}$.

$$x_{11} = p_{33}\left(p_{11}p_{33} - p_{22}^2\right)^{-1}, \;\; x_{31} = p_{22}\left(p_{11}p_{33} - p_{33}^2\right)^{-1}$$
$$x_{24} = x_{42} = p_{33}\left(p_{22}p_{44} - p_{22}^2\right)^{-1}.$$

4. THE PROPERTIES OF THE XIAO MATRICES

Lemma 2 [25]. *For completely controllable and observable MIMO LTI system* (2.1) *with a simple spectrum, the real matrices A, BB^{T}, for A with a simple spectrum, and distinct eigenvalues s_k, s_ρ, the following statements are true:*

1. *The matrix of the multiplier in the Hadamard decomposition of the controllability gramian P_c of the system is a solution of the Lyapunov equation*

$$AP_c + P_c A^{\mathrm{T}} = -I.$$

This matrix is the Xiao matrix.

2. *The multiplier matrix in the Hadamard decomposition of the controllability sub-gramian P_{ci} is a solution of the modal Lyapunov equation*

$$AP_{ci} + P_{ci}A^{\mathrm{T}} = -I_{ii}.$$

3. *The diagonal elements of the multiplier matrices are positive numbers*

$$p_{cii} = \sum_{k=1}^{n} \frac{1}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda=k}^{n} (-s_k - s_\lambda)} (-1)^{i-1} s_k^{2(i-1)} > 0, \qquad i = \overline{1,n}.$$

4. *The diagonal elements of the controllability sub-gramian matrices in the form of Xiao matrices form complex or real geometric progressions*

$$p_{c11i} = p_{c11i-1} q_i,$$

where first element is $p_{c111} = \frac{1}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda=k}^{n} (-s_k - s_\lambda)}$, *denominator is* $q_i = -s_i^2$, $i = \overline{1,n}$.

5. *The trace of the Xiao matrix is a positive number that represents the sum of the progressions formed by the diagonal terms of the controllability sub-gramians matrices*

$$\operatorname{tr} P_c = \sum_{i=1}^{n} \frac{1}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda=k}^{n} (-s_k - s_\lambda)} \frac{(s_i^2)^{i-1} - 1}{(-s_i^2 - 1)} > 0. \tag{4.1}$$

Proof. The validity of statement 1 follows from the Hadamard decomposition of the controllability gramian P_c of the form (2.8). It is proved in [24, 25] that this solution is singular and is the Xiao matrix. From [24] (Theorem 2, corollary 3) it follows that the matrix of the multiplier Ω in the Hadamard decomposition of the controllability gramian P_c has the form

$$P(t) = \Omega(t) \circ \Psi, \quad \Omega = \omega(n, s_k, -s_k, i, j) 1_{n\times n}, \quad 1_{n\times n} = \sum_{j=1}^{n} \sum_{\eta=1}^{n} e_j e_\eta^T,$$

which proves the first part of statement 1 of the lemma. On the other hand, the Xiao matrix is symmetric and real, which implies its normality. For such matrices there is a Schur transform that reduces them to triangular form. It is important that in this case, the eigenvalues of the matrices are located on the diagonal [11]. The validity of statement 2 follows from the fact that, when the conditions of the lemma are met, the equality is true

$$P_c = \sum_{i=1}^{n} P_{ci}.$$

The diagonal elements of the controllability sub-gramians matrices are determined by the formulas (2.8), from which the validity of statements 1–3 follows. The positivity of their sum follows from the condition of complete controllability and observability of the MIMO LTI system (statement 4). From this condition it follows that the trace of the Xiao matrix is a positive number and is the sum of progressions formed by the diagonal terms of the controllability sub-gramians matrices. From this follows the validity of the formula of the statement 5

$$\operatorname{tr} P_c = \sum_{i=1}^{n} \frac{1}{\dot{N}(s_i) N(-s_i)} \frac{(s_i^2)^{n-1} - 1}{(-s_i^2 - 1)} > 0.$$

Note that the representations of spectral decompositions of gramians in Hadamard form are closely related to geometric control theory [7]. Multiplier matrices arise naturally when considering Hadamard products, and the simplest representations of these matrices, as shown above, appear when using canonical representations in the form of controllability and observability. First of all, when describing matrices, it is necessary to distinguish significant and zero elements. Both have the property of periodicity of the structure, which allows to classify these matrices as pseudo-Hankel matrices. Significant elements always appear in the place of the main and secondary diagonals in the case when the sum of the indices of their row and column is an even number. In turn, zero elements always appear in the main and secondary diagonals in the case when the sum of the indices of their row and column is an odd number. Matrices of multipliers consisting of significant and zero elements are symmetric matrices. Any square matrix, including the matrix of the solution of the Lyapunov equation, can be represented in the separable form

$$P = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} \mathbf{1}_{ij},$$

where the matrix $\mathbf{1}_{ij}$ is the multiplier matrix, consisting of zeros, except for the element "ij", which is equal to one. The Hadamard transform is a structural transformation, which allows one to separate the scalar and matrix parts of the spectral decompositions of the gramians, in which the scalar part determines the matrix of multipliers, and the matrix part is associated with the matrices of the decomposition of the resolvent of the dynamics matrix in the Faddeev–Leverrier series and the transformed matrices of the right-hand sides of the Lyapunov equations. An important property of the multipliers of SISO LTI systems in the canonical forms of controllability and observability is their positive definiteness, a consequence of which is the positivity of the diagonal elements and the trace of the corresponding gramians. Their elements depend only on the eigenvalues of the dynamics matrix and its characteristic polynomial, which are independent of similarity transformations and, therefore, are invariants under these transformations. Unlike the multiplier matrices of SISO LTI systems, the matrix part of the spectral decompositions of MIMO LTI depends on similarity transformations, but in this case the use of invariant multiplier matrices allows to obtain closed formulas for evaluating any elements of the gramian matrices. From the general formulas for calculating gramians it follows that in this case the multiplier matrix is common for the controllability and observability gramians [25]. Note that the multiplier matrices for SISO LTI systems can be calculated using Routh tables using the coefficients of the characteristic equation, which can be calculated without calculating the eigenvalues of the dynamics matrix. The advantages of this approach are the possibility of obtaining closed formulas for calculating the direct and inverse controllability gramians for SISO LTI systems and the absence of the need to consider spectral decompositions in the case of multiple eigenvalues [11]. Diagonal canonical forms differ from canonical forms of controllability and observability in that for the former, gramians and sub-gramians are generally complex matrices, while for the latter, they are real.

5. SPECTRAL DECOMPOSITIONS OF SOLUTIONS OF LYAPUNOV DIFFERENTIAL EQUATIONS ON A FINITE INTERVAL

Then assume that system (2.1) is, unless otherwise stated, completely controllable, and observable. Consider the Lyapunov differential equation [25]

$$\frac{dP(t)}{dt} = AP\left(t\right) + P\left(t\right)A^{\mathrm{T}} + BB^{\mathrm{T}}, \quad P\left(0\right) = 0_{nxn}, \ t \in \left[0, T\right], \tag{5.1}$$

where BB^{T} is a real matrix of size $(n \times n)$.

A solution to this equation using operational calculus and the decomposition of the resolvent of the dynamics matrix A in the Faddeev–Leverrier series will be contracted. The latter have the form [21, 22]

$$(Is - A)^{-1} = \sum_{j=0}^{n} A_j s^j [N(s)]^{-1}, \quad A_j = \sum_{i=j+1}^{n} a_i A^{i-j+1},$$
$$(Is - A^{\mathrm{T}})^{-1} = \sum_{j=0}^{n} A_j^{\mathrm{T}} s^j [N(s)]^{-1},$$

where A_j, A_j^{T} are Faddeev matrices constructed for resolvent matrices using the Faddeev–Leverrier algorithm; $N(s)$ is characteristic polynomial of matrices A, A^{T}; a_i are coefficients of this polynomial.

The first method of spectral decompositions of solutions of Sylvester's differential equations is based on the lemma

Lemma 3 [24]. *For completely controllable and observable MIMO LTI system* (2.1) *with a simple spectrum, the real matrices A, BB^{T}, for A with a simple spectrum, and distinct eigenvalues s_k, s_ρ,*

$$s_k + s_\rho \neq 0, \quad k = \overline{1,n}; \quad \rho = \overline{1,n},$$

and for the solution of the Lyapunov equations on a finite half-interval $[0,t) \in [0,\ T]$, the system can be represented as

$$x_d = Tx, \quad \dot{x}_d = A_d x_d + B_d u, \quad y_d = C_d x_d,$$
$$A_d = TAT^{-1}, \quad B_d = TB, \quad C_d = CT^{-1}, \quad Q_d = TBB^{\mathrm{T}}T^{\mathrm{T}},$$

or

$$A = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_n \end{bmatrix} \begin{bmatrix} \nu_1^* \\ \nu_2^* \\ \vdots \\ \nu_n^* \end{bmatrix} = T\Lambda T^{-1},$$

where T is composed of right eigenvectors u_i, and T^{-1} is composed of left eigenvectors ν_i^*, corresponding to its own number s_i.

Then the controllability grammian of the diagonalized linear part is a solution of the Lyapunov equation, which is determined from the formula

$$P_d^c = \sum_{k=1}^{n} \sum_{\rho=1}^{n} p_{dk,\rho}^c e_k e_\rho^{\mathrm{T}},$$
$$A_d = \operatorname{diag}\{\dots s_k \dots\} = Q_1 A Q_1^{-1},$$

where Q_1 is matrix of size $n \times n$.

In this case, the solution of the Lyapunov differential equation on a finite half-interval $[0,t) \in [0,T]$ has the form

$$P_d^c(t) = \left[p_{dk,\rho}^c(t)\right],$$
$$p_{dk,\rho}^c(t) = \frac{r_{dk,\rho}^c e^{(s_k+\rho)t}}{s_k + s_\rho} + p_{dk,\rho}^c, \quad p_{dk,\rho}^c = -\frac{r_{dk,\rho}^c}{s_k + s_\rho}, \quad r_{dk,\rho}^c = e_k Q_1 BB^{\mathrm{T}}(Q_1^{\mathrm{T}}) e_\rho^{\mathrm{T}},$$
$$P^c(t) = Q_1^{-1} P_d(t) \left(Q_1^{\mathrm{T}}\right)^{-1}.$$

The second method for solving Lyapunov differential equations is based on the use of the Laplace transform to calculate the Lyapunov integral and the decomposition of the resolvent of the dynamics matrix in a Faddeev–Leverrier series.

Theorem 2. *For system* (5.1) *and satisfied conditions of Lemma 3, the following statements are true:*

The case of a simple combination spectrum of a matrix A

1. *Spectral decompositions of solutions of Lyapunov differential equations* (5.1) *in the form of Hadamard products for the combination spectrum of the dynamics matrices are*

$$P_{j\eta}(t) = \Omega_{j\eta}(t) \circ \Psi_{j\eta}, \quad \Psi_{j\eta} = A_j BB^{\mathrm{T}} B_\eta, \tag{5.2}$$

$$P_{j\eta}(t) = \sum_{k=1}^{n}\sum_{\rho=1}^{n} \frac{s_k^j {s_\rho}^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_\rho - s_\lambda)} \left[\frac{e^{(s_k+s_\rho)t}-1}{s_k+s_\rho}\right] A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}}, \tag{5.3}$$

$$\Omega_{j\eta}(t) = \sum_{k=1}^{n}\sum_{\rho=1}^{n} \frac{s_k^j {s_\rho}^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_\rho - s_\lambda)} \left[\frac{e^{(s_k+s_\rho)t}-1}{s_k+s_\rho}\right] e_j e_\eta^{\mathrm{T}},$$

$$P(t) = \Omega(t) \circ \Psi, \quad \Psi = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} A_j BB^{\mathrm{T}} B_\eta,$$

$$\Omega(t) = \sum_{k=1}^{n}\sum_{\rho=1}^{n}\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} \frac{s_k^j {s_\rho}^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_\rho - s_\lambda)} \left[\frac{e^{(s_k+s_\rho)t}-1}{s_k+s_\rho}\right] e_j e_\eta^{\mathrm{T}}.$$

2. *For the case of decomposition of solutions of Lyapunov differential equations for the simple spectrum of the dynamics matrix, the same formulas are valid* (5.2)–(5.3) *with other multiplier matrices*

$$P_{j\eta}(t) = \sum_{k=1}^{n} \frac{s_k^j (-s_k)^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda=\rho}^{n} (-s_\rho - s_\lambda)} \left(e^{s_k t} - 1\right) A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}} = \widetilde{\Omega}_{j\eta}(t) \circ \Psi_{j\eta}, \tag{5.4}$$

$$\widetilde{\Omega}_{j\eta}(t) = \sum_{k=1}^{n} \frac{s_k^j (-s_k)^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda=\rho}^{n} (-s_\rho - s_\lambda)} \left(e^{s_k t} - 1\right) e_j e_\eta^{\mathrm{T}}, \quad \Psi_{j\eta} = A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}}, \tag{5.5}$$

$$P(t) = \widetilde{\Omega}(t) \circ \Psi, \quad \Psi = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}}, \tag{5.6}$$

$$\Omega(t) = \sum_{k=1}^{n}\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}} (s_k - s_\lambda) \prod_{\lambda=1,\lambda=\rho}^{n} (-s_\rho - s_\lambda) \left(e^{s_k t} - 1\right) e_j e_\eta^{\mathrm{T}}. \tag{5.7}$$

3. *The Hermitian component of the spectral decompositions of solutions of the Lyapunov equations is*

$$P^H(t) = \frac{1}{2}\left(P(t) + P^*(t)\right), \qquad P_{j\eta}^H(t) = \frac{1}{2}\left(P_{\boldsymbol{j\eta}}(t) + P_{\boldsymbol{j\eta}}^*(t)\right), \tag{5.8}$$

where the spectral decompositions of the matrices $P, P^*, P_{\boldsymbol{j\eta}}, P^*_{\boldsymbol{j\eta}}$ *are determined by the formulas* (5.2)–(5.7)*.*

The case of a multiple spectrum of the matrix A

4. *Solution of Lyapunov differential equations* (5.1) *in the form of Hadamard products for a multiple spectrum of the dynamics matrices have the form*

$$P_{j\eta}(t) = \Omega_{j\eta}(t) \circ \Psi_{j\eta}, \quad \Psi_{j\eta} = A_j BB^{\mathrm{T}} B_\eta, \tag{5.9}$$

$$P_{j\eta}(t) = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} p_{cj\eta}(t) A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}},$$

$$p_{cj\eta}(t) = \mathcal{L}^{-1} \left\{ s^{-1} \sum_{\delta=1}^{n} \sum_{\rho=1}^{m_\delta} K_{\delta\rho j} \frac{(-1)^{m_\delta - \rho}}{(m_\delta - \rho)!} \left[\frac{d^{m_\delta-\rho}}{ds^{m_\delta-\rho}} \left(\frac{s^\eta}{\prod\limits_{\lambda=1,\lambda\neq\delta}^{n} (-s - s_\lambda)^{m_\lambda}} \right) \right]_{s=s-s_\delta} \right\},$$

$$K_{\delta\rho j} = \frac{1}{(\rho-1)!} \left[\frac{d^{\rho-1}}{ds^{\rho-1}} \left(\frac{s^j}{\prod\limits_{\lambda=1,\lambda\neq\delta}^{n} (-s - s_\lambda)^{m_\lambda}} \right) \right]_{s=s_\delta},$$

$$\Omega_{j\eta}(t) = p_{cj\eta}(t) e_j e_\eta^{\mathrm{T}},$$

$$P(t) = \Omega(t) \circ \Psi, \quad \Psi = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} A_j BB^{\mathrm{T}} B_\eta,$$

$$\Omega(t) = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} p_{cj\eta}(t) e_j e_\eta^{\mathrm{T}}. \tag{5.10}$$

5. *The Hermitian component of the solutions of the Lyapunov equations has the form* (5.8)*.*

Proof. The solution to the differential equation (5.1) is an integral of the form [1, 3, 25]

$$P(t) = \int_{o}^{\mathrm{T}} e^{A\tau} BB^{\mathrm{T}} e^{A^{\mathrm{T}}\tau} d\tau. \tag{5.11}$$

We apply the Laplace transform to both parts of the Lyapunov equation, considering the initial conditions to be zero and using the theorem on the Laplace transform of the product of real functions of time, the image of which is a fractional-rational algebraic fraction [26]. In the case under consideration, this fraction contains one zero pole, and all other poles are simple. Using the decomposition of the resolvent in the Faddeev–Leverrier series and substituting the obtained expressions into (5.11), the image of the decomposition of the solution of the Lyapunov differential equations (5.1) in the combination spectrum of the dynamics matrices is obtained

$$P(s) = \frac{1}{s} \sum_{k=1}^{n} \sum_{\rho=1}^{n} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \frac{-1}{s_k + s_\rho} \frac{s_k^j s_\rho{}^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_\rho - s_\lambda)} A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}}$$

$$+ \sum_{k=1}^{n} \sum_{\rho=1}^{n} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \frac{-1}{s_k + s_\rho} \frac{s_k^j s_\rho{}^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod\limits_{\lambda=1,\lambda\neq k}^{n} (s_\rho - s_\lambda)} A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}} \frac{1}{s - s_k - s_\rho}.$$

By performing the inverse transformation, the spectral decomposition of the solution of the Lyapunov differential equations (5.1) over the combination spectrum of the dynamics matrices in the time domain in the form of Hadamard products is obtained

$$P_{j\eta}(t)=\sum_{k=1}^{n}\sum_{\rho=1}^{n}\frac{s_k^j s_\rho{}^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k-s_\lambda)\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_\rho-s_\lambda)}\left[\frac{e^{(s_k+s_\rho)t}-1}{s_k+s_\rho}\right]A_jBB^{\mathrm{T}}A_\eta^{\mathrm{T}}=\Omega_{j\eta}(t)\circ\Psi_{j\eta},$$

$$\Omega_{j\eta}(t)=\sum_{k=1}^{n}\sum_{\rho=1}^{n}\frac{s_k^j s_\rho{}^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k-s_\lambda)\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_\rho-s_\lambda)}\left[\frac{e^{(s_k+s_\rho)t}-1}{s_k+s_\rho}e_je_\eta^{\mathrm{T}}\right],$$

$$\Psi_{j\eta}=A_jBB^{\mathrm{T}}A_\eta^{\mathrm{T}},$$

$$P(t)=\Omega(t)\circ\Psi,\quad \Psi=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}A_jBB^{\mathrm{T}}A_\eta^{\mathrm{T}}, \tag{5.12}$$

$$\Omega(t)=\sum_{k=1}^{n}\sum_{\rho=1}^{n}\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\frac{s_k^j s_\rho{}^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k-s_\lambda)\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_\rho-s_\lambda)}\left[\frac{e^{(s_k+s_\rho)t}-1}{s_k+s_\rho}\right]e_je_\eta^{\mathrm{T}}.$$

The equality (5.12) expresses the spectral decomposition of solutions of Lyapunov differential equations over the combination spectrum of the matrix. This proves the first statement of the theorem. Via the identity

$$\sum_{k=1}^{n}\sum_{\rho=1}^{n}\frac{-1}{s_k+s_\rho}\frac{s_k^j s_\rho{}^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k-s_\lambda)\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_\rho-s_\lambda)}\equiv\sum_{k=1}^{n}\frac{s_k^j(-s_k)^\eta}{\dot N(s_k)\,N(-s_k)}, \tag{5.13}$$

Similar decompositions in terms of the simple spectrum of the matrix A is obtained

$$P_{j\eta}(t)=\sum_{k=1}^{n}\frac{s_k^j(-s_k)^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k-s_\lambda)\prod\limits_{\lambda=1,\lambda=\rho}^{n}(-s_\rho-s_\lambda)}(e^{s_kt}-1)A_jBB^{\mathrm{T}}A_\eta^{\mathrm{T}}=\Omega_{j\eta}(t)\circ\Psi_{j\eta},$$

$$\Omega_{j\eta}(t)=\sum_{k=1}^{n}\frac{s_k^j(-s_k)^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k-s_\lambda)\prod\limits_{\lambda=1,\lambda=\rho}^{n}(-s_\rho-s_\lambda)}(e^{s_kt}-1)e_je_\eta^{\mathrm{T}},\ \ \Psi_{j\eta}=A_jBB^{\mathrm{T}}A_\eta^{\mathrm{T}},$$

$$P(t)=\Omega(t)\circ\Psi,\quad \Psi=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}A_jBB^{\mathrm{T}}A_\eta^{\mathrm{T}},$$

$$\Omega(t)=\sum_{k=1}^{n}\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\frac{s_k^j(-s_k)^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k-s_\lambda)\prod\limits_{\lambda=1,\lambda=\rho}^{n}(-s_\rho-s_\lambda)}(e^{s_kt}-1)e_je_\eta^{\mathrm{T}}.$$

The resulting decompositions prove the second statement of the theorem. The third statement follows from statements 1 and 2.

First, the calculation of the Lyapunov integral (5.7) has singularities in the case of multiple roots of the characteristic equation of the dynamics matrix. Due to the properties of the decomposition of the resolvent in the Faddeev–Leverrier series

$$(Is - A)^{-1} = \sum_{j=0}^{n-1} \frac{A_j s^j}{N(s)}, \quad (Is - A^{\mathrm{T}})^{-1} = \sum_{j=0}^{n-1} \frac{A_j^{\mathrm{T}} s^j}{N(s)}.$$

In accordance with the formula for the inverse Laplace transform for a fractional rational function

$$\mathcal{L}^{-1} \sum_{j=0}^{n-1} \frac{A_j s^j}{N(s)} = \sum_{\delta=1}^{n} \sum_{\rho=1}^{m_\delta} K_{\delta\rho j} t^{m_\delta - \rho} e^{s_\delta t},$$

$$K_{\delta\rho j} = \frac{1}{(\rho-1)!} \left[\frac{d^{\rho-1}}{ds^{\rho-1}} \left(\frac{\sum\limits_{j=0}^{n-1} A_j s^j}{\prod\limits_{\lambda=1, \lambda\neq\delta}^{n} (s - s_\lambda)^{m_\lambda}} \right) \right]_{s=s_\delta}.$$

The image of the Lyapunov integral using the obtained expressions and the theorem on the image of the product of two fractional rational functions of time, when one of the multiplied images has multiple poles [26], is

$$\mathcal{L}[P_C(t) = \mathcal{L} \left\{ \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} [p_{cj\eta}(t)] \, e_j e_\eta^{\mathrm{T}} \right\} \tag{5.14}$$

$$= s^{-1} \left\{ \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \sum_{\delta=1}^{n} \sum_{\rho=1}^{m_\delta} K_{\delta\rho j} \frac{(-1)^{m_\delta - \rho}}{(m_\delta - \rho)!} \left[\frac{d^{m_\delta - \rho}}{ds^{m_\delta - \rho}} \left(\frac{s^\eta}{\prod\limits_{\lambda=1, \lambda\neq\delta}^{n} (-s - s_\lambda)^{m_\lambda}} \right) \right]_{s=s-s_\delta} \right\}$$

$$\times A_j B B^{\mathrm{T}} A_\eta^{\mathrm{T}},$$

$$K_{\delta\rho j} = \frac{1}{(\rho-1)!} \left[\frac{d^{\rho-1}}{ds^{\rho-1}} \left(\frac{s^j}{\prod\limits_{\lambda=1, \lambda\neq\delta}^{n} (-s - s_\lambda)^{m_\lambda}} \right) \right]_{s=s_\delta}.$$

The formulas follow from this

$$\Omega_{j\eta}(t) = p_{cj\eta}(t) \, e_j e_\eta^{\mathrm{T}},$$

$$P(t) = \Omega(t) \circ \Psi, \quad \Psi = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} A_j B B^{\mathrm{T}} B_\eta,$$

$$\Omega(\mathrm{t}) = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \; p_{cj\eta}(t) \, e_j e_\eta^{\mathrm{T}}. \tag{5.15}$$

The equality (5.14) expresses the spectral decomposition of the solutions of the Lyapunov equations over the multiple spectrum of the matrix A. Consider an important special case of continuous linear stationary SISO LTI systems represented by state equations in canonical controllability and observability forms. In this case, the controllability and observability gramians in Hadamard form are determined by the formulas (2.6), (2.7)

$$P^F_{cj\eta}(t) = P^F_{oj\eta}(t) = \sum_{k=1}^{n} \frac{s_k^j(-s_k)^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k - s_\lambda)\prod\limits_{\lambda=1,\lambda=k}^{n}(-s_k - s_\lambda)}(e^{skt}-1)e_j e_\eta^{\mathrm{T}},$$

$$\widetilde{\Omega}_{j\eta}(t) = \sum_{k=1}^{n} \frac{s_k^j(-s_k)^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k - s_\lambda)\prod\limits_{\lambda=1,\lambda=k}^{n}(-s_k - s_\lambda)}(e^{skt}-1), \quad \Psi_{j\eta} = e_j e_\eta^{\mathrm{T}},$$

$$P^F_c(t) = \widetilde{\Omega}(t) \circ \Psi, \quad \Psi = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} e_j e_\eta^{\mathrm{T}},$$

$$\widetilde{\Omega}(t) = \sum_{k=1}^{n}\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} \frac{s_k^j(-s_k)^\eta}{\prod\limits_{\lambda=1,\lambda\neq k}^{n}(s_k - s_\lambda)\prod\limits_{\lambda=1,\lambda=k}^{n}(-s_k - s_\lambda)}(e^{skt}-1)e_j e_\eta^{\mathrm{T}}.$$

The multiplier of the final sub-gramian of controllability is directly proportional to the residue of the transfer function of the system, multiplied by the value of the transfer function of the anti-stable system when substituting the value of the root into it s_k [27].

Corollary 1. *If the conditions of Theorem 2 be satisfied, then the limit at infinity of the spectral decompositions of the solutions of the Lyapunov differential equations* (2.3) *in the form of Hadamard products for the multiple spectrum of the dynamics matrices has the form*

$$P_{j\eta}(\infty) = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} p_{cj\eta}(\infty)\, A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}},$$

$$\Omega_{j\eta}(\infty) = p_{cj\eta}(\infty)\, e_j e_\eta^{\mathrm{T}},$$

$$p_{cj\eta}(\infty) = \left\{ \sum_{\delta=1}^{n}\sum_{\rho=1}^{m_\delta} K_{\delta\rho j}\frac{(-1)^{m_\delta-\rho}}{(m_\delta-\rho)!}\left[\frac{d^{m_\delta-\rho}}{ds^{m_\delta-\rho}}\left(\frac{s^\eta}{\prod\limits_{\lambda=1,\lambda\neq\delta}^{n}(-s-s_\lambda)^{m_\lambda}}\right)\right]_{s=-s_\delta}\right\},$$

$$P(\infty) = \Omega(\infty) \circ \Psi, \quad \Psi = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} A_j BB^{\mathrm{T}} B_\eta,$$

$$\Omega(\infty) = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} p_{cj\eta}(\infty)\, e_j e_\eta^{\mathrm{T}}.$$

Proof. The transition to the limit at infinity in the expression (5.14) is completed. In accordance with the finite value theorem

$$p_{cj\eta}(\infty) = \left\{ \sum_{\delta=1}^{n}\sum_{\rho=1}^{m_\delta} K_{\delta\rho j}\frac{(-1)^{m_\delta-\rho}}{(m_\delta-\rho)!}\left[\frac{d^{m_\delta-\rho}}{ds^{m_\delta-\rho}}\left(\frac{s^\eta}{\prod\limits_{\lambda=1,\lambda\neq\delta}^{n}(-s-s_\lambda)^{m_\lambda}}\right)\right]_{s=-s_\delta}\right\},$$

$$P_{j\eta}(\infty) = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} p_{cj\eta}(\infty)\, A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}},$$

$$\Omega_{j\eta}(\infty) = p_{cj\eta}(\infty)\, e_j e_\eta^{\mathrm{T}}.$$

Thus, expressions for the limit at infinity of spectral decompositions of solutions of Lyapunov differential equations (3.3) in the form of Hadamard products are obtained

$$P(\infty) = \Omega(\infty) \circ \Psi, \quad \Psi = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} A_j B B^{\mathrm{T}} B_\eta,$$

$$\Omega(\infty) = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} p_{cj\eta}(\infty)\, e_j e_\eta^{\mathrm{T}}.$$

Formulas for spectral decompositions of solutions of Lyapunov algebraic equations in the form of Hadamard products for multiple eigenvalues are obtained.

Corollary 2. *For system* (5.1) *the spectral decomposition of the finite gramian of observability in the form of Hadamard products has the form of formulas* (5.2)–(5.8), (5.9)–(5.10), *in which the matrix* $A_j B B^{\mathrm{T}} A_\eta^{\mathrm{T}}$ *replaced by matrix* $A_j^{\mathrm{T}} C^{\mathrm{T}} C A_\eta$. *The multipliers of finite controllability and observability grammians coincide on a half-interval* $[0, t) \in [0,\ T]$. *Statements 3 and 4 coincide. The proof of Corollary 2 coincides with the proof of Corollary 1 in accordance with the substitution.*

Corollary 3. *For system* (2.1), *satisfied conditions of Lemma 2 and Corollary 2, defined controllability* P_c *and observability* P_o *infinite gramians and simple spectra of singular values of the matrices of gramians* σ_i *exist a transformation of the variables of the system* (2.1) *with matrices* T_{c2} *and* T_{o2} *of the form*

$$x_{cd} = T_{c2} x, \quad x_{od} = T_{o2} x,$$

in which the gramians acquire a diagonal appearance

$$T_{c2} P_c\, T_{c2}^{-1} = P_{cd}, \quad P_{cd} = \operatorname{diag}\left\{ \sigma_{c1} \quad \sigma_{c2} \quad \ldots \quad \sigma_{cn} \right\}, \tag{5.16}$$

$$T_{o2} P_c\, T_{o2}^{-1} = P_{od}, \quad P_{od} = \operatorname{diag}\left\{ \sigma_{o1} \quad \sigma_{o2} \quad \ldots \quad \sigma_{on} \right\}, \tag{5.17}$$

where T_{c2} *is matrix composed of the right eigenvectors of the gramian* P_c, *and* T_{c2}^{-1} *is matrix composed of the left eigenvectors of the gramian* P_c, T_{o2} *is matrix composed of the right eigenvectors of the gramian* P_o, *and* T_{o2}^{-1} *is matrix composed of the left eigenvectors of the gramian* P_o.

Then the following spectral decompositions are true

$$\operatorname{tr} P_{cd} = \sum_{i=1}^{n} \sigma_{ci}, \quad \operatorname{tr} P_{od} = \sum_{i=1}^{n} \sigma_{oi},$$

$$\operatorname{tr} P_{cd}^{-1} = \left[p_{c0}^{-1} p_{c1} \sum_{i=1}^{n} \frac{1}{\dot{N}_c(\sigma_{ci})} + p_{c0}^{-1} p_{c2} \sum_{i=1}^{n} \frac{\sigma_{ci}}{\dot{N}_c(\sigma_{ci})} + \cdots + p_{c0}^{-1} \sum_{i=1}^{n} \frac{\sigma_{ci}^{n-1}}{\dot{N}_c(\sigma_{ci})} \right] > 0, \tag{5.18}$$

$$\operatorname{tr} P_{od}^{-1} = \left[p_{o0}^{-1} p_{o1} \sum_{i=1}^{n} \frac{1}{\dot{N}_o(\sigma_{ci})} + p_{o0}^{-1} p_{o2} \sum_{i=1}^{n} \frac{\sigma_{oi}}{\dot{N}_o(\sigma_{oi})} + \cdots + p_{o0}^{-1} \sum_{i=1}^{n} \frac{\sigma_{oi}^{n-1}}{\dot{N}_o(\sigma_{oi})} \right] > 0,$$

Proof. The statements will be proved for the case of controllability gramians.

The decomposition of the resolvent of the controllability gramian P_{cd} in the Faddeev–Leverrier series

$$(I\sigma - P_{cd})^{-1} = \sum_{i=1}^{n} \frac{P_{cd,n-1}\, \sigma_{ci}^{n-1} + \cdots + P_{cd,1}\sigma_{ci} + P_{cd,0}}{\dot{N}_c(\sigma_{ci})} \frac{1}{\sigma - \sigma_i}, \tag{5.19}$$

in which P_{cdj} are Faddeev matrices for the gramian decomposition, calculated using the Faddeev–Leverrier algorithm in the form

$$P_{cdj} = p_{cd,j+1}I + p_{cdj+2}P_{cd} \ + \dots + P_{cd}^{n-j-1},$$

where p_{cdj} are coefficients of the characteristic equation of the controllability gramian matrix P_{cd}. Due to the diagonality of the matrices P_{cd}, the matrix P_{cdj} can be written in the form

$$P_{cdj} = \begin{bmatrix} \sum\limits_{j=0}^{n-1} p_{cd,n-j}\sigma_{cd1}^{j} & 0 & \dots & 0 \\ 0 & \sum\limits_{j=0}^{n-1} p_{cd,n-j}\sigma_{cd2}^{j} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \sum\limits_{j=0}^{n-1} p_{cd,n-j}\sigma_{cdn}^{j} \end{bmatrix}. \tag{5.20}$$

From (3.2)

$$P_{cd}^{-1} = -p_{cd0}^{-1}R_1 = p_{cd0}^{-1}\left[-p_{cd1}I - p_{cd2}P_{cd} - \dots - P_{cd}^{n-1}\right],$$

$$\operatorname{tr} P_{cd}^{-1} = \left[p_{c0}^{-1}p_{c1}\sum_{i=1}^{n}\frac{1}{\dot N_c(\sigma_{ci})} + p_{c0}^{-1}p_{c2}\sum_{i=1}^{n}\frac{\sigma_{ci}}{\dot N_c(\sigma_{ci})} + \dots + p_{c0}^{-1}\sum_{i=1}^{n}\frac{\sigma_{ci}^{n-1}}{\dot N_c(\sigma_{ci})}\right].$$

This proves the first statement of the investigation.. The positivity of the trace follows from the positivity of the singular values σ_{ci}. The initial controllability gramian is determined from the formula

$$P_c = T_{c2}^{-1}P_{cd}T_{c2}. \tag{5.21}$$

The proof of the statement for the case of observability gramians repeats the proof for the case of controllability gramians.

Continuous stationary SISO LTI systems represented by state equations in canonical controllability and observability forms is considered. The first step in transforming equations of the form (2.1) into canonical controllability form consists of transforming a system of the form (2.1)

$$x = R_c^F x_c,$$
$$\dot x_c(t) = A_c^F x_c(t) + b_c^F u(t), \quad x_c(0) = 0, \ \ y_c^F(t) = c_c^F x_c(t), \quad k = 0, 1, 2, \dots,$$
$$A_c^F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad b_c^F = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^{\mathrm T}.$$

The controllability gramians at this stage are Xiao matrices

$$P_c^F = \sum_{k=1}^{n}\sum_{\eta=0}^{n-1}\sum_{j=0}^{n-1}\frac{s_k^j(-s_k)^\eta}{\dot N(s_k)\,N(-s_k)}1_{j+1\eta+1}.$$

Corollary 4. *In some cases, it is necessary to transform the obtained equations of state into a new form, which transforms the controllability gramians into a diagonal form*

$$x_d = T_d x_c, \quad \dot x_d = P_{cd}x_d, \quad P_{cd} = \operatorname{diag}\left\{ \sigma_{c1} \ \ \sigma_{c2} \ \ \dots \ \ \sigma_{cn} \right\}.$$

The new variables are related to the variables of the original system by the equation

$$x_d = T_d x_c = T_d (R_c^F)^{-1} x. \tag{5.22}$$

It follows that the similarity transformation matrix T at the second stage is equal to

$$T = T_d (R_c^F)^{-1}.$$

It is obvious that the controllability gramians of the original and transformed systems and their inverse gramians are related by the relations

$$P_c = T^{-1} P_{cd} (T^{-1})^{\mathrm{T}} = R_c^F T_{dc}^{-1} P_{cd} (T_{dc}^{-1})^{\mathrm{T}} (R_c^F)^{\mathrm{T}},$$
$$P_c^{-1} = T^{-1} P_{cd}^{-1} (T^{-1})^{\mathrm{T}} = R_c^F T_{dc}^{-1} P_{cd}^{-1} (T_{dc}^{-1})^{\mathrm{T}} (R_c^F)^{\mathrm{T}}.$$

6. FORMULAS OF SPECTRAL DECOMPOSITIONS OF SOME ENERGY METRICS

Statement. *Consider MIMO LTI systems (2.1) in the canonical controllability form. Assume that these systems are stable, matrices A, B are real, have a simple spectrum, their eigenvalues s_k, s_ρ are different, do not belong to the imaginary axis of the eigenvalue plane, and the conditions are satisfied*

$$s_k + s_\rho \neq 0, \quad k = \overline{1,n}; \quad \rho = \overline{1,n_1}; \quad s_\rho, s_\rho \in spec\ A.$$

Then the following decompositions of the energy metrics of the controllability grammians over the spectrum of the dynamics matrix A are valid

1. $Vol\,\epsilon_x = c_n \sqrt{Det\left[\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{k=1}^{n} \omega(n, s_k, j, \eta)\, A_j B B^{\mathrm{T}} (A_\eta)^{\mathrm{T}}\right]}, \quad c_n = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}.$

2. $Vol\,\epsilon_y = c_n \sqrt{Det\left[\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{k=1}^{n} \omega(n, s_k, j, \eta) C A_j B B^{\mathrm{T}} (A_\eta)^{\mathrm{T}} C^{\mathrm{T}}\right]}, \ c_n = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}.$

3. $\mathrm{tr}\, P_c = \omega(n, s_k, 0, 0)\, \mathrm{tr}\left[A_0 B B^{\mathrm{T}} (A_0)^{\mathrm{T}}\right] + \omega(n, s_k, 1, 1)\, \mathrm{tr}\left[A_1 B B^{\mathrm{T}} (A_1)^{\mathrm{T}}\right] + \ldots$
$+ \omega(n, s_k, n-1, n-1)\, \mathrm{tr}\left[A_{n-1} B B^{\mathrm{T}} (A_{n-1})^{\mathrm{T}}\right].$

4. SISO LTI : $\mathrm{tr}\, P_c^F = \omega(n, s_k, 0, 0) + \omega(n, s_k, 1, 1) + \cdots + \omega(n, s_k, n-1, n-1)$.

5. $E_{in} = \dfrac{1}{2} x^{\mathrm{T}} P^{-1} c x,$ (6.1)

$$P_c^{-1} = \sum_{i=1}^{n} \frac{P_{cd,n-1}\sigma_{ci}^{n-1} + \cdots + P_{cd,1}\sigma_{ci} + P_{cd,0}}{\dot{N}_c(\sigma_{ci})} \frac{1}{\sigma_{ci}}, \ P_{cdj} = p_{cd,j+1} I + p_{cd,j+2} P_{cd} + \cdots + P_{cd}^{n-j-1}.$$

6. $E_{out} = \dfrac{1}{2} x^{\mathrm{T}} P_0 x,$ (6.2)

$$\text{MIMO LTI}: \ P_0 = \left[\sum_{k=1}^{n}\sum_{\eta=0}^{n-1}\sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} 1_{j+1\eta+1}\right] \circ \left[\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} A_j B B^{\mathrm{T}} B_\eta\right],$$

$$\text{SISO LTI}: \ P_0^F = \sum_{k=1}^{n}\sum_{\eta=0}^{n-1}\sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} 1_{j+1\eta+1}.$$

7. *Index of centrality of energy metrics of controllability of continuous multi-connected stationary systems* J_{CE}

$$J_{CE} = \operatorname{tr} P_c = \sum_{i=1}^{n} \frac{1}{\dot{N}(s_i) N(-s_i)} \frac{\left(s_i^2\right)^{i-1} - 1}{\left(-s_i^2 - 1\right)}. \tag{6.3}$$

8. *Average minimum energy* [19]

$$E_{avmin} = \frac{1}{n}\left[p_{c0}^{-1} p_{c1} \sum_{i=1}^{n} \frac{1}{\dot{N}_c(\sigma_{ci})} + p_{c0}^{-1} p_{c2} \sum_{i=1}^{n} \frac{\sigma_{ci}}{\dot{N}_c(\sigma_{ci})} + \cdots + p_{c0}^{-1} \sum_{i=1}^{n} \frac{\sigma_{ci}^{n-1}}{\dot{N}_c(\sigma_{ci})}\right].$$

Proof of Statement. Decompositions 1 and 2 follow from the known formulas for calculating the volume of the attraction ellipsoid [6], into which the expressions for the spectral decompositions of the controllability and observability gramians are substituted. In this form, the formulas allow to estimate the influence of the eigenvalues of the dynamics matrix on the volume of the attraction ellipsoid. Statement 3 follows from formulas (5.6) and (5.7). Decomposition 4 follows from formulas (2.6) and (2.7). The validity of decomposition 5 on the degree of reachability of the network (6.1) follows from [3], formulas (5.18) and (5.20) of Corollary 3. The validity of decomposition 6 follows from [3], formulas (5.6) and (5.7) of Theorem 2. The centrality indices of the energy metrics of controllability of individual modes of continuous multiconnected stationary systems $J_{CE,}$, as shown above, can be determined through the trace of the Xiao matrix. The latter is the sum of geometric progressions of individual modes in accordance with formula (4.1) of Lemma 2. Decomposition 8 follows from [19] and formula (5.18) of Corollary 3. Note that the Xiao matrices play a major role in calculating most of the energy metrics considered above. Energy metrics play an important role in the stability analysis of linear systems. (6.2), (6.3) show a negative synergy of interaction of weakly stable modes: the closer individual modes are to each other, the greater the energy accumulated in the group of modes, the closer the system is to the stability boundary. The classical criterion of the degree of stability, based on the distance from the nearest root of the characteristic equation to the imaginary axis, does not reveal such synergy.

7. CONCLUSION

The use of transformations of the equations of state into canonical forms of controllability and observability allowed one to simplify the formulas for spectral decompositions of the matrices of gramians. In the article, both the spectra of the system dynamics matrix and the spectra of the singular values of the gramians are considered as spectra. In the article, new spectral and structural decompositions of finite gramians in Hadamard form are obtained for solutions of algebraic and differential Lyapunov equations of linear stationary multi-connected systems with many inputs and many outputs, including the case of multiple roots of the characteristic equation of the system. In this case, the scalar part of Hadamard products depends on time, and the matrix part depends on the Faddeev matrices in the decomposition of the resolvent of the dynamics matrix and the right-hand sides of the Lyapunov equations. It is shown that the scalar function of the multiplier is an invariant under similarity transformations and strongly depends on the difference between the eigenvalues of the dynamics matrix and their multiplicity. It forms the main energy metrics of the basic energy balance of the system. The matrix part of Hadamard products forms the weight coefficients in the spectral decomposition of the square of the H_2-norm of the transfer function of the dynamics matrix. The obtained results are generalized for the class of dynamic networks. In this case, finite gramians, which are the solution of the Lyapunov differential matrix equations, play an important role. Formulas (2.1)–(2.8) provide the key to solving the optimization problem (2.3) and add the ability to control the current stability reserves. The degree of network stability is determined using the energy metric of the square of the H_2-norm of the transfer function of the dynamics matrix, which makes it possible not only to establish the fact of dissipativity of transient processes, but also to study the degree of their attenuation for weakly stable oscillatory systems or systems with multiple roots of the characteristic equation [26]. The degree of controllability of the network is associated with the minimum energy, which allows to introduce new metrics of energy efficiency of control in the form of a quadratic form formed by its inverse controllability gramian, obtained using spectral decompositions. Much attention is paid to invariant energy metrics formed using Xiao matrices. The work uses known energy metrics of dynamic networks and develops methods and algorithms for their spectral decomposition as an additional tool for their analysis and optimization. The results obtained can be used to design modal control systems and solve problems of optimal placement of sensors and actuators in control systems.

REFERENCES

1. Antoulas, A.C., *Approximation of Large-Scale Dynamical Systems*, Philadelphia: SIAM, 2005.
2. Gantmakher, F.R., *Teoriya matrits* (Theory of Matrices), Moscow: Nauka, 1966. Translated into English under the title *Theory of Matrices*, New York: Chelsea, 1959.
3. Benner, P. and Damm, T., Lyapunov equations, Energy Functionals and Model Order Reduction of Bilinear and Stochastic Systems, *SIAM J. Control Optim.*, 2011, vol. 49, pp. 686–711.
4. Zubov, N.E., Zybin, E.Yu., Mikrin, E.A., Misrikhanov, M.Sh., and Ryabchenko, V.N., General analytical forms of solutions of Sylvester and Lyapunov equations for continuous and discrete dynamic systems, *Izvestiya RAN, Teoriya i sistemy upravleniya*, 2017, no. 1, pp. 3–20.
5. Ikramov, H.D., *Chislennoe reshenie matrichnykh uravnenii* (Numerical solution of the matrix equations), Moscow: Nauka, 1984.
6. Polyak, B.T., Khlebnikov, M.V., and Rapoport, L.B., *Teoriya avtomaticheskogo upravleniya* (Automatic control theory), Moscow: LENAND, 2019.
7. Sachkov, Yu.L., *Upravlyaemost' i simmetrii invariantnykh sistem na gruppakh Li i odnorodnykh prostranstvakh* (Controllability of invariant systems on lie groups and homogeneous spaces), Moscow: Fizmatlit, 2006.
8. Liu, Y., Slotine, J., and Barabasi, A., Controllability of complex networks, *Nature*, 2011, vol. 473, pp. 167–173.
9. Xiao, C.S., Feng, Z.M., and Shan, X.M., On the Solution of the Continuous-Time Lyapunov Matrix Equation in Two Canonical Forms, *IEE Proc.*, 1992, vol. 139, no. 3, pp. 286–290.
10. Mehr, F., *A Determination of Design of Optimal Actuator Location Based on Control Energy*, London: University of London, 2018.
11. Hauksdottir, A. and Sigurdsson, S., The continuous closed form controllability gramian and its inverse, *2009 American Control Conference Hyatt Regency Riverfront, St. Louis, MO, USA June 10–12*, 2009, pp. 5345–5351.
12. Sreeram, V. and Agathoklis, P., Solution of Lyapunov equation with system matrix in companion form, *IEE Proc. D. Control. Theory Appl*, 1991, vol. 138, no. 6, pp. 529–534.
13. Dilip, A.S.A., The controllability gramian, the Hadamard product and the optimal actuator, *Leader Sensor Select. Problem Nature Phys*, 2015, vol. 11, pp. 779–786.
14. Mironovskii, L.A. and Solov'eva, T.N., Analysis and synthesis of modally balanced systems, *Autom. Remote Control*, 2013, vol. 74, no. 4, pp. 588–603.
15. Pasqualetti, F., Zampieri, S., and Bullo, F., Controllability metrics, limitations and algorithms for complex networks, *IEEE Transact. Control Network Syst.*, 2014, vol. 1, no. 1, pp. 40–52.
16. Zheleznov, K.O. and Khlebnikov, M.V., Application of the invariant ellipsoid method to solve a linear tracking problem, *Works of MIPT*, 2013, vol. 5, no. 4, pp. 115–121.
17. Lindmark, G. and Altafini, C., Minimum energy control for complex networks, *Sci. Reports*, 2018, vol. 8, no. 3188.

18. Poolla, B.K., Bolognani, S., and Dorfler, F., Optimal Placement of Virtual Inertia in Power Grids, *IEEE Transact. Autom. Control*, 2017, vol. 62, no. 12, pp. 6209–6220.

19. Summers, T., Cortesi, F., and Lygeros, J., On submodularity and controllability in complex dynamical networks, *IEEE Transact. Control Network Syst.*, 2015, vol. 3, no. 1, pp. 91–101.

20. Yadykin, I.B., Spectral Decompositions of gramians of Continuous Stationary Systems Given by Equations of State in Canonical Forms, *Mathematics*, 2022, vol. 10, no. 13, pp. 2339.

21. Lindmark, G. and Altafini, C., Combining centrality measures for control energy reduction in network controllability problems, *Proc. 2019 European Control Conference (ECC)*, 2019, pp. 1518–1523.

22. Faddeev, D.K. and Faddeeva, V.N., *Vychislitel'nye metody lineinoi algebry* (Computational methods of linear algebra), Moscow: Lan', 2009.

23. Hanson, B. and Peeters, R., A Faddeev Sequence Method for solving Lyapunov and Sylvester Equations, *Linear Algebra Appl.*, 1996, vol. 241–243, pp. 401–430.

24. Yadykin, I.B. and Galyaev, A.A., On the methods for calculation of grammians and their use in analysis of linear dynamic systems, *Autom. Remote Control*, 2013, vol. 74, no. 2, pp. 207–224.

25. Yadykin, I.B. and Galyaev, I.A., Spectral decompositions of grammians and energy metrics of continuous unstable control systems, *Autom. Remote Control*, 2023, vol. 84, no. 10, pp. 1243–1258.

26. Iskakov, A. and Yadykin, I., Lyapunov modal analysis and participation factors applied to small-signal stability of power systems, *Automatica*, 2021, vol. 132, no. 109814.

27. Bahtadze, N.N., Chereshko, A.A., Elpashev, D.V., Yadykin, I.B., Sabitov, R.A., and Smirnova, G.S., Associative Model Predictive Control, *IFAC PapersOnLine. Yokohama, Japan: Elsevier*, 2023, vol. 56, no. 2, pp. 7330–7334.

28. Gardner, M.F. and Barns, J.L., Transients in linear systems studied by the Laplace transformation, *vol. 1. Lumped-constant systems. New York, London. Wiley, Chapman and Hall*, 1942.

This paper was recommended for publication by A.A. Galyaev, a member of the Editorial Board